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BIFURCATION CONDITIONS FOR A THICK ELASTIC PLATE UNDER THRUST

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Abstract—A thick rectangular plate of incompressible isotropic elastic material is subjected to a pure homogeneous deformation by tensile forces or thrusts applied to a pair of opposite faces. The theory of small deformations superposed on finite deformations is applied to determine the critical conditions under which bifurcation solutions (i.e. adjacent equilibrium positions) can exist. The adjacent equilibrium positions considered are those for which the superposed deformation is two-dimensional and is coplanar with the loading force and the thickness direction of the plate, the faces of the plate normal to its thickness being force-free. A number of theorems relating to the critical conditions for superposed deformations of the flexural and barreling types are derived under conditions on the strain-energy function more general than those employed in earlier work. It is also shown how these results can be applied to the determination of the bifurcation conditions corresponding to *any* specified strain-energy function.

1. INTRODUCTION

Following the formulation of the theory of the superposition of infinitesimal deformations on finite deformations of elastic bodies by Green *et al.* [1], an extensive literature has evolved in which this theory is applied to the calculation of critical loading conditions for instability of elastic bodies of various shapes. The theory is used to determine the load for which non-zero superposed deformations first become possible. It is assumed in all of these calculations that this loading condition will, in fact, correspond to the onset of instability. We shall call this assumption, which is used also in the present paper, the *assumption of exchange of stabilities*.†

The first papers in which the theory of Green *et al.*[1] was used to obtain critical conditions for instability appear to be those of Wilkes[2], and Green and Spencer[3]. Wilkes discussed the instability under thrust of a thick circular tube and of a circular cylinder with respect to radially symmetrical buckling and Green and Spencer discussed the instability of a circular cylinder under simultaneous torsion and tension. Fosdick and Shield[4] have discussed the instability of a circular cylinder under thrust, with respect to flexural instability. Guo Zhong-Heng[5] has considered both flexural and barreling instability (i.e. antisymmetric and symmetric instability) of a thick circular plate under radial thrust. While the secular equation, which determines the critical bifurcation conditions, is obtained for a general incompressible isotropic elastic material, he discusses the implications of the equation only for certain special forms of the strain-energy function.

† This is in partial accord with the terminology commonly employed in the study of stability problems in fluid mechanics. There, the term *principle of exchange of stabilities* is usually used, but we prefer the more descriptive word *assumption* to the more pretentious *principle*.

In the present paper, we discuss the critical bifurcation conditions for a thick rectangular plate of incompressible isotropic elastic material. Taking the surfaces of the plate to be perpendicular to the axes of a rectangular Cartesian coordinate system x, we consider that the plate is initially subjected to an arbitrary pure homogeneous deformation, with principal directions parallel to the axes of the system x, its surfaces normal to the x_2-axis being force-free. The existence is investigated of neighboring solutions in which a deformation in the x_1x_2-plane is superposed on the pure homogeneous deformation, the distance between the surfaces normal to the x_1-axis remaining fixed and these surfaces being free of tangential traction in the x_2-direction, and the plate is held with its x_3-dimension fixed.

Let λ_1, λ_2, λ_3 be the principal extension ratios in the x_1, x_2 and x_3 directions respectively and let $W = W(I_1, I_2)$ be the strain-energy function per unit volume, where I_1 and I_2 are the usual first and second strain-invariants of finite elasticity theory, defined in terms of the λ's by equation (2.15). It is seen in this paper† that the critical bifurcation conditions depend on W through the dependence on the λ's of a quantity A defined by

$$A = \frac{2(\lambda_1 + \lambda_2)^2}{W_1 + \lambda_3^2 W_2}(W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22}),$$

where

$$W_A = \partial W/\partial I_A, \quad W_{AB} = \partial^2 W/\partial I_A\,\partial I_B \qquad (A, B = 1, 2).$$

If λ_3 is fixed, it follows from the incompressibility condition $\lambda_1\lambda_2\lambda_3 = 1$ that the λ's are uniquely determined if $\lambda_2/\lambda_1 = \lambda$, say, is specified. Hence, for fixed λ_3, we can regard A as a function of λ.

Denoting the dimensions of the plate parallel to the x_1 and x_2 axes, in its undeformed state, by $2l_1$ and $2l_2$ respectively, the critical bifurcation conditions depend on the aspect ratio l_2/l_1 through a quantity η defined by $n\pi l_2/l_1$ or $(n - 1/2)\pi l_2/l_1$, accordingly as the deformation is symmetric or antisymmetric with respect to the x_2x_3-plane, where n determines the number of half wave-lengths parallel to the x_1-direction in the deformation. In the former case there are $2n$ such half-waves and in the latter case $2n - 1$.

It has been shown elsewhere[6] that the material is inherently unstable if $A < -(\lambda + 1)^2/(\lambda - 1)^2$ and it appears likely, although this has not so far been proven, that this is also true if $-1 > A > -(\lambda + 1)^2/(\lambda - 1)^2$. We accordingly restrict our discussion to the case $A \geq -1$. It is shown that, subject to this limitation, if A is constant, then for each value of η there are two values of λ corresponding to critical bifurcation conditions. The lesser of these values corresponds to flexural deformations and the greater to barreling deformations (see Figs. 1 and 2). Also, for each constant A, η increases monotonically to infinity as the critical value of λ for flexural deformations increases from 1 to some limiting value which itself increases with A. For higher values of λ than this limiting value only barreling deformations are possible and η decreases monotonically as the critical value of λ increases to infinity. It is also shown how, from the results, the critical bifurcation conditions for arbitrary dependence of A on λ can be obtained. Furthermore, it has been shown that, with $A \geq -1$, the critical bifurcation conditions cannot be satisfied for any value of λ less than unity, i.e. for tensile conditions.

† An analogous result was obtained for the circular plate under uniform radial thrust by Guo Zhong-Heng[5].

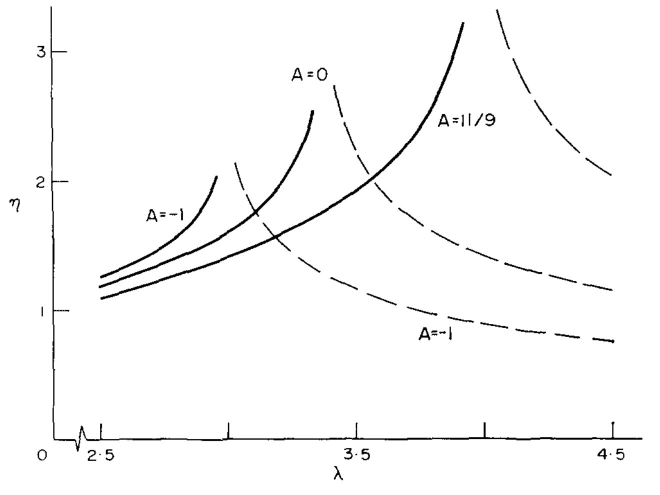

Fig. 1. η vs critical λ curves for three constant values of A.

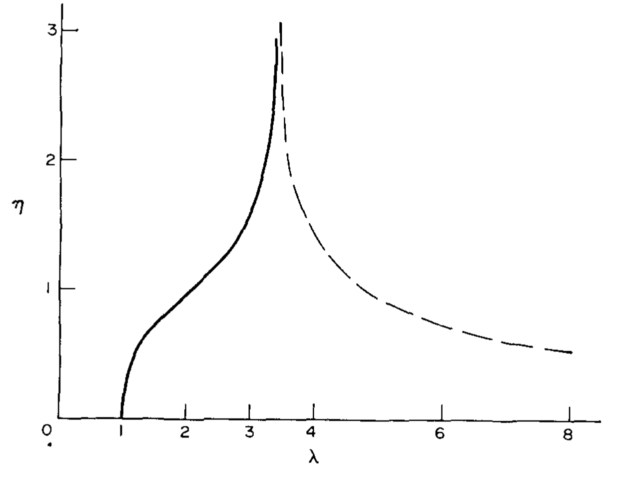

Fig. 2. η vs critical λ curve for $A = 0$.

2. BASIC EQUATIONS

We consider the deformation of a body of incompressible isotropic elastic material. In the deformation, a generic particle of the body, which, at some fixed reference time, is at ξ_α in the coordinate system x, moves to x_i at time t. The deformation is completely described if the dependence of x_i on ξ_α and t,

$$x_i = x_i(\xi_\alpha, t), \tag{2.1}$$

is known. The Finger strain tensor $\bar{C}_{ij}$ associated with this deformation is given by†

$$\bar{C}_{ij} = x_{i,\alpha} x_{j,\alpha}. \tag{2.2}$$

† The usual summation convention applies to lower case Latin and Greek subscripts. The comma notation $,\alpha$ is used to denote $\partial/\partial\xi_\alpha$.

We define three scalar invariants $\bar{I}_A (A = 1, 2, 3)$ of this tensor by

$$\bar{I}_1 = \bar{C}_{ii}, \qquad \bar{I}_2 = \tfrac{1}{2}[(\bar{C}_{ii})^2 - \bar{C}_{ij}\bar{C}_{ji}], \qquad \bar{I}_3 = |\bar{C}_{ij}|. \tag{2.3}$$

Since, for an incompressible material all deformations are necessarily isochoric, we have

$$\bar{I}_3 = 1. \tag{2.4}$$

For an incompressible isotropic elastic material, the strain-energy W per unit volume must depend on the deformation gradients $x_{i,\alpha}$ through $\bar{I}_1$ and $\bar{I}_2$ only, so that

$$W = W(\bar{I}_1, \bar{I}_2). \tag{2.5}$$

The Cauchy stress tensor $\bar{\Sigma}_{ij}$ associated with the deformation (2.1) is then given by

$$\bar{\Sigma}_{ij} = 2[(\bar{W}_1 + \bar{I}_1 \bar{W}_2)\bar{C}_{ij} - \bar{W}_2 \bar{C}_{ik}\bar{C}_{kj}] - \bar{P}\,\delta_{ij}, \tag{2.6}$$

where $\bar{P}$ is an arbitrary hydrostatic pressure, δ_{ij} denotes the Kronecker delta and we have introduced the notation

$$\bar{W}_A = \partial W / \partial \bar{I}_A \qquad (A = 1, 2). \tag{2.7}$$

The Piola–Kirchoff stress tensor $\bar{\Pi}_{\alpha i}$ is then given by

$$\bar{\Pi}_{\alpha i} = \frac{\partial \xi_\alpha}{\partial x_j} \bar{\Sigma}_{ij}. \tag{2.8}$$

Substitution of (2.6) in (2.8) then yields

$$\bar{\Pi}_{\alpha i} = 2[(\bar{W}_1 + \bar{I}_1 \bar{W}_2)x_{i,\alpha} - \bar{W}_2 x_{i,\beta} x_{k,\beta} x_{k,\alpha}] - \tfrac{1}{2}\bar{P}\varepsilon_{ipq}\varepsilon_{\alpha\beta\gamma} x_{p,\beta} x_{q,\gamma}, \tag{2.9}$$

where ε_{ijk} is the alternating symbol.

In the absence of body forces, the Piola–Kirchoff equations of motion are

$$\bar{\Pi}_{\alpha i,\alpha} = \rho \ddot{x}_i, \tag{2.10}$$

where ρ is the mass density of the material and a dot denotes differentiation with respect to time.

Let ν_α be the unit normal to the surface of the body in the reference configuration and $\bar{T}_i$ be the surface traction at time t, measured per unit area in this configuration. Then,

$$\bar{T}_i = \bar{\Pi}_{\alpha i} \nu_\alpha. \tag{2.11}$$

We now suppose that the deformation (2.1) consists of a small time-dependent deformation, superposed on a finite static pure homogeneous deformation with extension ratios $\lambda_A (A = 1, 2, 3)$. Then, we may write†

$$x_A = \lambda_A \xi_A + \varepsilon u_A \qquad (A = 1, 2, 3), \tag{2.12}$$

where the λ's are positive constants, $u_i = u_i(\xi_\alpha, t)$ and ε is a small constant, the square and higher powers of which may be uniformly neglected. Substitution from (2.12) in (2.2) yields

$$\bar{C}_{ij} = C_{ij} + \varepsilon c_{ij}, \quad \bar{I}_A = I_A + \varepsilon i_A \qquad (A = 1, 2, 3), \tag{2.13}$$

† The summation convention does not apply to upper case Latin subscripts.

where

$$C_{AA} = \lambda_A^2, \quad C_{AB} = 0 \qquad (A \neq B; \quad A, B = 1, 2, 3)$$

and (2.14)

$$c_{AB} = \lambda_A u_{B,A} + \lambda_B u_{A,B} \qquad (A, B = 1, 2, 3).$$

Also, from (2.3), (2.13) and (2.14), we obtain, with (2.4),

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 = 1 \tag{2.15}$$

and

$$\begin{aligned} i_1 &= 2(\lambda_1 u_{1,1} + \lambda_2 u_{2,2} + \lambda_3 u_{3,3}), \\ i_2 &= 2\{\lambda_1(\lambda_2^2 + \lambda_3^2)u_{1,1} + \lambda_2(\lambda_3^2 + \lambda_1^2)u_{2,2} + \lambda_3(\lambda_1^2 + \lambda_2^2)u_{3,3}\}, \\ i_3 &= 2\left(\frac{1}{\lambda_1}u_{1,1} + \frac{1}{\lambda_2}u_{2,2} + \frac{1}{\lambda_3}u_{3,3}\right) = 0. \end{aligned} \tag{2.16}$$

We expand $\overline{W}_1$ and $\overline{W}_2$ in a Taylor series about (I_1, I_2) and, neglecting terms of $0(\varepsilon^2)$, we write

$$\begin{aligned} \overline{W}_1 &= W_1 + \varepsilon(W_{11}i_1 + W_{12}i_2), \\ \overline{W}_2 &= W_2 + \varepsilon(W_{21}i_1 + W_{22}i_2), \end{aligned} \tag{2.17}$$

where

$$W_A = \overline{W}_A\Big|_{\bar{I}_1, \bar{I}_2 = I_1, I_2} \quad \text{and} \quad W_{AB} = \frac{\partial^2 W}{\partial \bar{I}_A\, \partial \bar{I}_B}\Big|_{\bar{I}_1, \bar{I}_2 = I_1, I_2} \qquad (A, B = 1, 2). \tag{2.18}$$

Also we write

$$\overline{P} = P + \varepsilon p \quad \text{and} \quad \overline{\Pi}_{\alpha i} = \Pi_{\alpha i} + \varepsilon \pi_{\alpha i}. \tag{2.19}$$

Introducing (2.13), (2.17), (2.18) and (2.19) into (2.9), we obtain

$$\begin{aligned} \Pi_{AA} &= 2\lambda_A[W_1 + (I_1 - \lambda_A^2)W_2] - \frac{P}{\lambda_A}, \\ \Pi_{AB} &= 0 \qquad (A \neq B), \end{aligned} \tag{2.20}$$

and†

$$\begin{aligned} \pi_{AA} &= \sum_{C=1}^{3} K_{AACC} u_{C,C} - \frac{p}{\lambda_A}, \\ \pi_{AB} &= K_{ABAB} u_{A,B} + K_{ABBA} u_{B,A}, \qquad (A \neq B) \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} K_{AABB} &= 2[W_1 + (I_1 - \lambda_A^2)W_2]\delta_{AB} + \frac{1 - \delta_{AB}}{\lambda_A\lambda_B}[-P + 4\lambda_A^2\lambda_B^2 W_2] \\ &\quad + 4\lambda_A\lambda_B[W_{11} + (2I_1 - \lambda_A^2 - \lambda_B^2)W_{12} + (I_1 - \lambda_A^2)(I_1 - \lambda_B^2)W_{22}], \\ K_{ABBA} &= K_{BAAB} = 2[W_1 + (I_1 - \lambda_A^2 - \lambda_B^2)W_2], \qquad (A \neq B) \\ K_{ABAB} &= K_{BABA} = \frac{1}{\lambda_A\lambda_B}(P - 2\lambda_A^2\lambda_B^2 W_2). \qquad (A \neq B). \end{aligned} \tag{2.22}$$

† We use the notation $_{,A}$ to denote $\partial/\partial\xi_A$.

Introduction of (2.12) and (2.19) into the equations of motion (2.10) yields

$$\Pi_{\alpha i,\alpha} = 0 \quad \text{and} \quad \pi_{\alpha i,\alpha} = \rho \ddot{u}_i . \tag{2.23}$$

Again, writing

$$\bar{T}_i = T_i + \varepsilon t_i , \tag{2.24}$$

we obtain from (2.11) and (2.19)

$$T_i = \Pi_{\alpha i} \nu_\alpha \quad \text{and} \quad t_i = \pi_{\alpha i} \nu_\alpha . \tag{2.25}$$

3. DEFORMATION OF A RECTANGULAR BLOCK

We now apply the relations derived in the previous section to a body which is initially a rectangular block located with its edges parallel to the axes of the reference system x. Let $2l_A$ be the initial length of the edge which is parallel to the A-axis, so that in the undeformed state the block occupies the region

$$-l_A \le \xi_A \le l_A . \tag{3.1}$$

We consider only problems in which the surface tractions on the surface initially normal to the 2-axis vanish, so that

$$T_i = t_i = 0 \quad \text{for} \quad \xi_2 = \pm l_2 . \tag{3.2}$$

Also, we restrict ourselves to the case when the superposed displacements u_i are in the 12-plane and are independent of ξ_3, so that

$$u_3 = 0 \quad \text{and} \quad u_A = u_A(\xi_1, \xi_2, t) \qquad (A = 1, 2). \tag{3.3}$$

From (3.2) and (2.25) we see that on the surfaces $\xi_2 = \pm l_2$, for which $\nu_\alpha = \pm\delta_{\alpha 2}$, we have

$$T_i = \pm\Pi_{2i} = 0 \quad \text{and} \quad t_i = \pm\pi_{2i} = 0. \tag{3.4}$$

With (2.20) and (3.4), we obtain

$$P = 2\lambda_2^2[W_1 + (I_1 - \lambda_2^2)W_2] \tag{3.5}$$

and

$$\begin{aligned} \Pi_{11} &= \frac{2}{\lambda_1}(\lambda_1^2 - \lambda_2^2)(W_1 + \lambda_3^2 W_2), \\ \Pi_{33} &= \frac{2}{\lambda_3}(\lambda_3^2 - \lambda_2^2)(W_1 + \lambda_1^2 W_2), \\ \Pi_{22} &= 0, \quad \Pi_{\alpha i} = 0 \qquad (\alpha \ne i). \end{aligned} \tag{3.6}$$

These expressions for $\Pi_{\alpha i}$ automatically satisfy equation $(2.23)_1$.

Introducing (3.3) into (2.21), we obtain

$$\begin{aligned} \pi_{AA} &= \sum_{C=1}^{2} K_{AACC} u_{C,C} - \frac{p}{\lambda_A} \qquad (A = 1, 2, 3), \\ \pi_{12} &= K_{1212} u_{1,2} + K_{1221} u_{2,1}, \\ \pi_{21} &= K_{2112} u_{1,2} + K_{2121} u_{2,1}, \\ \pi_{13} &= \pi_{31} = \pi_{23} = \pi_{32} = 0. \end{aligned} \tag{3.7}$$

Also, with (3.3), the incompressibility condition expressed by the last of equations (2.16) becomes

$$\frac{1}{\lambda_1} u_{1,1} + \frac{1}{\lambda_2} u_{2,2} = 0. \tag{3.8}$$

Introducing (3.7) into (2.23) and using (3.8), we obtain

$$\begin{aligned} (1 + a_1)u_{1,11} + u_{1,22} - \lambda_2 \hat{p}_{,1} &= k^2 \ddot{u}_1, \\ u_{2,11} + (1 + a_2)u_{2,22} - \lambda_1 \hat{p}_{,2} &= k^2 \ddot{u}_2, \\ \hat{p}_{,3} &= 0, \end{aligned} \tag{3.9}$$

where we have introduced the notation

$$\begin{aligned} a_1 &= \frac{1}{K_{2112}} \left[K_{1111} - \frac{\lambda_2}{\lambda_1}(K_{1122} + K_{2121})\right] - 1 \\ &= \frac{2(\lambda_1^2 - \lambda_2^2)}{W_1 + \lambda_3^2 W_2} [W_{11} + (\lambda_2^2 + 2\lambda_3^2)W_{12} + \lambda_3^2(\lambda_3^2 + \lambda_2^2)W_{22}], \\ a_2 &= \frac{1}{K_{1221}} \left[K_{2222} - \frac{\lambda_1}{\lambda_2}(K_{1212} + K_{2211})\right] - 1 \\ &= \frac{2(\lambda_2^2 - \lambda_1^2)}{W_1 + \lambda_3^2 W_2} [W_{11} + (\lambda_1^2 + 2\lambda_3^2)W_{12} + \lambda_3^2(\lambda_3^2 + \lambda_1^2)W_{22}], \\ \hat{p} &= \frac{\lambda_3 p}{2(W_1 + \lambda_3^2 W_2)} \quad \text{and} \quad k = \left\{\frac{\rho}{2(W_1 + \lambda_3^2 W_2)}\right\}^{1/2}. \end{aligned} \tag{3.10}$$

By using (3.8) and (3.7) and introducing the condition (3.2) that the incremental surface tractions vanish on the surfaces $\xi_2 = \pm l_2$, we obtain with (3.4), (3.5) and (2.22)

$$\left.\begin{aligned} \pm t_1 &= \pi_{21} = 2(W_1 + \lambda_3^2 W_2)(u_{1,2} + \lambda u_{2,1}) = 0 \\ \pm t_2 &= \pi_{22} = 2(W_1 + \lambda_3^2 W_2)[(2 + a_2)u_{2,2} - \lambda_1 \hat{p}] = 0 \end{aligned}\right\} \quad \text{on} \quad \xi_2 = \pm l_2, \tag{3.11}$$

where

$$\lambda = \lambda_2/\lambda_1. \tag{3.12}$$

We shall consider that on the faces $\xi_1 = \pm l_1$, for which $v_\alpha = \pm\delta_{\alpha 1}$, the displacement associated with the superposed deformation is zero in the 1-direction and the tangential traction in the 2-direction is zero, i.e.

$$u_1 = 0 \quad \text{and} \quad t_2 = \pi_{12} = 0 \quad \text{on} \quad \xi_1 = \pm l_1. \tag{3.13}$$

Then, from (2.25), (3.3) and (3.7), we obtain

$$u_1 = 0, \quad u_{2,1} = 0 \quad \text{on} \quad \xi_1 = \pm l_1. \tag{3.14}$$

4. SOLUTION OF THE EQUATIONS

Since, in obtaining stability criteria, we consider that the assumption of exchange of stabilities is valid, we shall obtain solutions of (3.9) and (3.8), subject to the boundary conditions (3.11) and (3.14), in the case when the displacements u_i are quasistatic, i.e. $\ddot{u}_i = 0$. In this case (3.9) and (3.8) become

$$\begin{aligned} (1 + a_1)u_{1,11} + u_{1,22} - \lambda_2 \hat{p}_{,1} &= 0, \\ u_{2,11} + (1 + a_2)u_{2,22} - \frac{\lambda_2}{\lambda} \hat{p}_{,2} &= 0, \end{aligned} \tag{4.1}$$

and

$$\lambda u_{1,1} + u_{2,2} = 0$$

respectively, where $\hat{p}$ is independent of ξ_3 and λ is given by (3.12).

We may obtain solutions of these equations in the forms

$$u_1 = \left.\begin{matrix} -\sin \Phi\xi_1 \\ \cos \Psi\xi_1 \end{matrix}\right\} U_1(\xi_2), \quad u_2 = \left.\begin{matrix} \cos \Phi\xi_1 \\ \sin \Psi\xi_1 \end{matrix}\right\} U_2(\xi_2), \quad \hat{p} = \left.\begin{matrix} \cos \Phi\xi_1 \\ \sin \Psi\xi_1 \end{matrix}\right\} \hat{P}(\xi_2), \tag{4.2}$$

where Φ and Ψ are constants. We note that two separate forms of solutions are given in (4.2), corresponding to the upper and lower terms. The upper terms correspond to symmetric and the lower terms to antisymmetric modes with respect to the 23-plane.

Substitution of (4.2) in (4.1) yields, in either case

$$\begin{aligned} U_1'' - (1 + a_1)\Omega^2 U_1 &= \lambda_2 \Omega \hat{P}, \\ (1 + a_2)U_2'' - \Omega^2 U_2 &= \frac{\lambda_2}{\lambda} \hat{P}', \\ -\lambda\Omega U_1 + U_2' &= 0, \end{aligned} \tag{4.3}$$

where a prime denotes differentiation with respect to ξ_2 and $\Omega = \Phi$ or Ψ accordingly as solutions given by the upper or lower terms in (4.2) are required. Eliminating $\hat{P}$ and U_1 from (4.3), we obtain

$$U_2^{(iv)} - [1 + \lambda^2 + A(1 - \lambda)^2]\Omega^2 U_2'' + \lambda^2\Omega^4 U_2 = 0, \tag{4.4}$$

where the notation

$$A = \frac{2(\lambda_1 + \lambda_2)^2}{W_1 + \lambda_3^2 W_2}(W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22}) \tag{4.5}$$

and the relation

$$a_1 + \lambda^2 a_2 = A(1 - \lambda)^2 \tag{4.6}$$

have been used.

The general solution of equation (4.4) is

$$U_2 = L_1 \cosh \Omega_1\xi_2 + L_2 \sinh \Omega_1\xi_2 + M_1 \cosh \Omega_2 \xi_2 + M_2 \sinh \Omega_2 \xi_2, \tag{4.7}$$

where

$$\Omega_1^2, \Omega_2^2 = \tfrac{1}{2}\Omega^2\{[1 + \lambda^2 + A(1 - \lambda)^2] \pm [\{1 + \lambda^2 + A(1 - \lambda)^2\}^2 - 4\lambda^2]^{1/2}\}, \tag{4.8}$$

provided that $\Omega_1^2 \neq \Omega_2^2$. The special case when $\Omega_1^2 = \Omega_2^2$ will be discussed later in Section 6.

Introducing (4.7) into $(4.3)_3$, we obtain

$$U_1 = \frac{1}{\lambda\Omega}[\Omega_1(L_1 \sinh \Omega_1\xi_2 + L_2 \cosh \Omega_1\xi_2) + \Omega_2(M_1 \sinh \Omega_2 \xi_2 + M_2 \cosh \Omega_2 \xi_2)]. \tag{4.9}$$

Again, introducing (4.9) into $(4.3)_1$, we obtain

$$\hat{P} = \frac{1}{\lambda_2 \lambda \Omega^2}[\Omega_1\{\Omega_1^2 - (1 + a_1)\Omega^2\}(L_1 \sinh \Omega_1\xi_2 + L_2 \cosh \Omega_1\xi_2) + \Omega_2\{\Omega_2^2 - (1 + a_1)\Omega^2\}(M_1 \sinh \Omega_2 \xi_2 + M_2 \cosh \Omega_2 \xi_2)]. \tag{4.10}$$

Introducing (4.2) into the boundary conditions (3.11) and (3.14), we obtain

$$\left.\begin{aligned} U_1' + \lambda\Omega U_2 &= 0 \\ (2 + a_2)U_2' - \lambda_1\hat{P} &= 0 \end{aligned}\right\} \quad \text{on} \quad \xi_2 = \pm l_2 \tag{4.11}$$

and

$$\left.\begin{matrix} -\sin \Phi\xi_1 \\ \cos \Psi\xi_1 \end{matrix}\right\} U_1 = 0, \qquad \left.\begin{matrix} -\Phi \sin \Phi\xi_1 \\ \Psi \cos \Psi\xi_1 \end{matrix}\right\} U_2 = 0 \quad \text{on} \quad \xi_1 = \pm l_1. \tag{4.12}$$

Ruling out the trivial case when $U_1 \equiv U_2 \equiv 0$, we obtain from (4.12),

$$\Phi = n\pi/l_1, \quad \Psi = (n - \tfrac{1}{2})\pi/l_1 \qquad (n = 1, 2, \ldots). \tag{4.13}$$

Eliminating U_1 from $(4.11)_1$ and $(4.3)_3$, we obtain

$$U_2'' + \lambda^2\Omega^2 U_2 = 0 \quad \text{on} \quad \xi_2 = \pm l_2. \tag{4.14}$$

Also, eliminating $\hat{P}$ and U_1 from $(4.11)_2$ and $(4.3)_{1,3}$, and using (4.6), we obtain

$$U_2''' - B\Omega^2 U_2' = 0 \quad \text{on} \quad \xi_2 = \pm l_2, \tag{4.15}$$

where

$$B = 1 + 2\lambda^2 + A(1 - \lambda)^2. \tag{4.16}$$

We note from (4.8) that

$$\Omega_1^2 + \lambda^2\Omega^2 = B\Omega^2 - \Omega_2^2 \quad \text{and} \quad \Omega_2^2 + \lambda^2\Omega^2 = B\Omega^2 - \Omega_1^2. \tag{4.17}$$

Introducing (4.7) into (4.14) and (4.15), we obtain, with (4.17),

$$(\Omega_1^2 + \lambda^2\Omega^2)(L_1 \cosh \Omega_1 l_2 \pm L_2 \sinh \Omega_1 l_2) + (\Omega_2^2 + \lambda^2\Omega^2)(M_1 \cosh \Omega_2 l_2 \pm M_2 \sinh \Omega_2 l_2) = 0,$$

and (4.18)

$$\Omega_1(\Omega_2^2 + \lambda^2\Omega^2)(L_1 \sinh \Omega_1 l_2 \pm L_2 \cosh \Omega_1 l_2) + \Omega_2(\Omega_1^2 + \lambda^2\Omega^2)(M_1 \sinh \Omega_2 l_2 \pm M_2 \cosh \Omega_2 l_2) = 0.$$

Equations (4.18) represent the four equations obtained by taking the upper signs together and the lower signs together. From these equations, we obtain

$$\begin{aligned}(\Omega_1^2 + \lambda^2\Omega^2)L_1 \cosh \Omega_1 l_2 + (\Omega_2^2 + \lambda^2\Omega^2)M_1 \cosh \Omega_2 l_2 = 0,\\ (\Omega_1^2 + \lambda^2\Omega^2)L_2 \sinh \Omega_1 l_2 + (\Omega_2^2 + \lambda^2\Omega^2)M_2 \sinh \Omega_2 l_2 = 0,\\ \Omega_1(\Omega_2^2 + \lambda^2\Omega^2)L_1 \sinh \Omega_1 l_2 + \Omega_2(\Omega_1^2 + \lambda^2\Omega^2)M_1 \sinh \Omega_2 l_2 = 0,\\ \Omega_1(\Omega_2^2 + \lambda^2\Omega^2)L_2 \cosh \Omega_1 l_2 + \Omega_2(\Omega_1^2 + \lambda^2\Omega^2)M_2 \cosh \Omega_2 l_2 = 0.\end{aligned} \tag{4.19}$$

From these equations, we see that either

$$L_1 = M_1 = 0 \quad \text{and} \quad \frac{\tanh \Omega_1 l_2}{\tanh \Omega_2 l_2} = \left(\frac{\Omega_2^2 + \lambda^2\Omega^2}{\Omega_1^2 + \lambda^2\Omega^2}\right)^2 \frac{\Omega_1}{\Omega_2}, \tag{4.20}$$

or

$$L_2 = M_2 = 0 \quad \text{and} \quad \frac{\tanh \Omega_2 l_2}{\tanh \Omega_1 l_2} = \left(\frac{\Omega_2^2 + \lambda^2\Omega^2}{\Omega_1^2 + \lambda^2\Omega^2}\right)^2 \frac{\Omega_1}{\Omega_2}, \tag{4.21}$$

unless

$$\frac{\tanh \Omega_1 l_2}{\tanh \Omega_2 l_2} = \frac{\tanh \Omega_2 l_2}{\tanh \Omega_1 l_2}, \tag{4.22}$$

i.e. unless

$$\sinh(\Omega_1 \pm \Omega_2) l_2 = 0. \tag{4.23}$$

This corresponds to the case when $\Omega_1^2 = \Omega_2^2$ which is discussed in Section 6.

We note from (4.19) that if (4.20) is applicable

$$\frac{L_2}{M_2} = -\frac{\Omega_2^2 + \lambda^2\Omega^2}{\Omega_1^2 + \lambda^2\Omega^2}\frac{\sinh \Omega_2 l_2}{\sinh \Omega_1 l_2} \tag{4.24}$$

and if (4.21) is applicable

$$\frac{L_1}{M_1} = -\frac{\Omega_2^2 + \lambda^2\Omega^2}{\Omega_1^2 + \lambda^2\Omega^2}\frac{\cosh \Omega_2 l_2}{\cosh \Omega_1 l_1}. \tag{4.25}$$

From (4.8) it is seen that three distinct cases arise accordingly as Ω_1^2 and Ω_2^2 are both positive, or complex conjugates, or both negative. It is easily seen that the latter case arises if and only if

$$A \leq -\left(\frac{\lambda + 1}{\lambda - 1}\right)^2. \tag{4.26}$$

It has been shown elsewhere[6] that if (4.26) applies then the material is inherently unstable and, accordingly, we need not discuss this case further. Also from (4.8), it is seen that Ω_1^2 and Ω_2^2 are complex conjugates if and only if

$$-\left(\frac{\lambda + 1}{\lambda - 1}\right)^2 < A < -1. \tag{4.27}$$

We shall postpone consideration of this case to a later paper and restrict the discussion in the present paper to the case when Ω_1^2 and Ω_2^2 are both positive.

Secular equations analogous to (4.20) and (4.21) were obtained by Guo Zhong-Heng[5] in his discussion of the instability of a circular plate under radial thrust. Secular equations similar to (4.20) and (4.21) have also been obtained, for a body having the same geometry as that discussed in the present paper, by Wesolowski[7], Levinson[8], Nowinski[9] and by Wu and Widera[10]. However, Levinson[8] derives these equations only for the particular case of an incompressible neo-Hookean material with $\lambda_2 = \lambda_3$. Nowinski[9], while deriving the equations for a general incompressible isotropic elastic material, does so only in the case when $\lambda_3 = 1$ and limits his discussion of the implications of the equations, with respect to critical bifurcation conditions, to the case when the material is neo-Hookean. Wu and Widera[10] derive the secular equations for the case when $\lambda_3 = 1$ and the material has a strain-energy function of the Mooney–Rivlin form. We note that in both the Mooney–Rivlin and neo-Hookean cases, $A = 0$. Burgess and Levinson[11] also derive equations analogous to (4.20) and (4.21) for two forms of the strain-energy function applicable to compressible materials. Wesolowski[7] derives the secular equations with the generality of the present paper, but limits his discussion of their implications to the tensile case ($\lambda < 1$) and concludes that instability may occur in tension for certain forms of the strain-energy function which yield complex forms for Ω_1 and Ω_2. It seems doubtful that such materials do, in fact, exist.

Guo Zhong-Heng[5] limits the discussion of the implications of his secular equations to three specific forms of the strain-energy function—the Mooney–Rivlin form (for which $A = 0$), a particular case when $A = -1$ and a particular case when $A = -(\lambda+1)^2/(\lambda-1)^2$. In the latter case Ω_1 and Ω_2 are equal and imaginary. It can be readily inferred from the results of Sawyers and Rivlin[6] that a strain-energy function for which $A = -(\lambda+1)^2/(\lambda-1)^2$ does not correspond to a real material. Later workers have invariably restricted the discussion of their secular equations to particular forms of the strain-energy function.

In the following sections, we discuss the implications of the secular equations (4.20) and (4.21) for arbitrary constant A, not less than -1, and show how from these results corresponding implications can be drawn for arbitrary dependence of A on λ, provided that A is not less than -1 for all relevant values of λ. In order to draw these conclusions, we first recast the secular equations into new forms (equations (5.11) and (5.12) below) more appropriate for our purposes.

5. DISCUSSION OF THE SECULAR EQUATION WHEN Ω_1^2 AND Ω_2^2 ARE POSITIVE

From (4.8), it is seen that this case arises if, and only if,

$$A > -1. \tag{5.1}$$

(Here we have omitted the case $\Omega_1^2 = \Omega_2^2$ in accord with our earlier remarks.) With the notation

$$\eta = l_2\Omega, \quad \Gamma_1 = \Omega_1/\Omega \quad \text{and} \quad \Gamma_2 = \Omega_2/\Omega, \tag{5.2}$$

we obtain, from (4.20),

$$\frac{\tanh\Gamma_1\eta}{\tanh\Gamma_2\eta} = \left(\frac{\Gamma_2^2+\lambda^2}{\Gamma_1^2+\lambda^2}\right)^2\frac{\Gamma_1}{\Gamma_2} \tag{5.3}$$

and, from (4.21),

$$\frac{\tanh\Gamma_2\eta}{\tanh\Gamma_1\eta} = \left(\frac{\Gamma_2^2+\lambda^2}{\Gamma_1^2+\lambda^2}\right)^2\frac{\Gamma_1}{\Gamma_2}. \tag{5.4}$$

If λ and λ_3 are given and the dependence of W on I_1 and I_2 is known, we may regard each of these equations as an equation for the determination of η. The value so determined is the critical value for which, with the assumption of exchange of stabilities, instability occurs at the specified values of λ and λ_3, in the mode considered. We note that, since in the cases covered by equation (5.3) $L_1 = M_1 = 0$, from (4.7) and (4.9), U_1 and U_2 are respectively even and odd functions of ξ_2. Accordingly, the modes covered by equation (5.3) are called *barreling modes*. Again, in the cases covered by equation (5.4) $L_2 = M_2 = 0$ and U_1 and U_2 are odd and even functions respectively of ξ_2. The modes covered by equation (5.4) are therefore called *flexural modes*.

The secular equations (5.3) and (5.4) may be rewritten in a somewhat different form. We write

$$\Gamma_1^2 = \lambda(\cosh 2\delta + \sinh 2\delta) \quad \text{and} \quad \Gamma_2^2 = \lambda(\cosh 2\delta - \sinh 2\delta), \tag{5.5}$$

where, from (4.8),

$$\cosh 2\delta = \frac{1}{2\lambda}[1 + \lambda^2 + A(1-\lambda)^2]$$

and (5.6)

$$\sinh 2\delta = \frac{1}{2\lambda}[\{1 + \lambda^2 + A(1-\lambda)^2\}^2 - 4\lambda^2]^{1/2}.$$

Whence,

$$\cosh^2 \delta = \frac{1}{4\lambda}[(1+\lambda)^2 + A(1-\lambda)^2]$$

and (5.7)

$$\sinh^2 \delta = \frac{1}{4\lambda}(1+A)(1-\lambda)^2.$$

From (5.5), we have

$$\Gamma_1 = \lambda^{1/2}(\cosh \delta + \sinh \delta) \quad \text{and} \quad \Gamma_2 = \lambda^{1/2}(\cosh \delta - \sinh \delta). \tag{5.8}$$

Introducing (5.8) into (5.3), we obtain

$$\frac{\sinh(2\lambda^{1/2}\cosh\delta)\eta + \sinh(2\lambda^{1/2}\sinh\delta)\eta}{\sinh(2\lambda^{1/2}\cosh\delta)\eta - \sinh(2\lambda^{1/2}\sinh\delta)\eta} = \frac{[\cosh 3\delta + (2\lambda+\lambda^2)\cosh\delta] - [\sinh 3\delta + (2\lambda-\lambda^2)\sinh\delta]}{[\cosh 3\delta + (2\lambda+\lambda^2)\cosh\delta] + [\sinh 3\delta + (2\lambda-\lambda^2)\sinh\delta]}. \tag{5.9}$$

Whence,

$$\frac{\sinh(2\lambda^{1/2}\cosh\delta)\eta}{\sinh(2\lambda^{1/2}\sinh\delta)\eta} = -\frac{\cosh 3\delta + (2\lambda+\lambda^2)\cosh\delta}{\sinh 3\delta + (2\lambda-\lambda^2)\sinh\delta}. \tag{5.10}$$

We thus have, for the barreling modes,

$$\frac{\sinh(2\lambda^{1/2}\cosh\delta)\eta}{\sinh(2\lambda^{1/2}\sinh\delta)\eta} = -\frac{\cosh\delta[(1+\lambda)^2 + 4\sinh^2\delta]}{\sinh\delta[-(1-\lambda)^2 + 4\cosh^2\delta]}. \tag{5.11}$$

In a similar manner, we obtain from (5.4), for the flexural modes

$$\frac{\sinh(2\lambda^{1/2}\cosh\delta)\eta}{\sinh(2\lambda^{1/2}\sinh\delta)\eta}=\frac{\cosh\delta[(1+\lambda)^2+4\sinh^2\delta]}{\sinh\delta[-(1-\lambda)^2+4\cosh^2\delta]}. \tag{5.12}$$

The solution of (5.12) for η is given by the intersection of the two curves

$$\Lambda=\frac{1}{\cosh\delta}[-(1-\lambda)^2+4\cosh^2\delta]\sinh(2\lambda^{1/2}\cosh\delta)\eta \tag{5.13}$$

and

$$\Lambda=\frac{1}{\sinh\delta}[(1+\lambda)^2+4\sinh^2\delta]\sinh(2\lambda^{1/2}\sinh\delta)\eta \tag{5.14}$$

in the Λ–η plane.

We note from (5.7) that

$$\cosh^2\delta>\sinh^2\delta. \tag{5.15}$$

It follows that, for real positive η,

$$\frac{1}{\cosh\delta}\sinh(2\lambda^{1/2}\cosh\delta)\eta>\frac{1}{\sinh\delta}\sinh(2\lambda^{1/2}\sinh\delta)\eta. \tag{5.16}$$

Also, for $\lambda\le 1$,

$$-(1-\lambda)^2+4\cosh^2\delta\ge(1+\lambda)^2+4\sinh^2\delta. \tag{5.17}$$

Consequently, for $\lambda\le 1$, the expression on the right-hand side of (5.13) is greater than that on the right-hand side of (5.14) for all real positive η and accordingly (5.12) has no real positive solutions for η.

If, however, $\lambda>1$, it can easily be seen that there is one real positive solution of (5.12) for η if

$$-(1-\lambda)^2+4\cosh^2\delta>0 \tag{5.18}$$

and no real positive solutions if

$$-(1-\lambda)^2+4\cosh^2\delta\le 0. \tag{5.19}$$

We note that in the latter case Λ, given by (5.13), is non-positive for η positive, while Λ, given by (5.14), is positive for η positive. Thus, there are no real solutions of (5.12) in the case (5.19). On the other hand, if the relation (5.18) applies, since for $\lambda>1$

$$-(1-\lambda)^2+4\cosh^2\delta<(1+\lambda)^2+4\sinh^2\delta, \tag{5.20}$$

the slope of (5.13) at the origin is less than that of (5.14). However, in view of (5.16) and (5.18), at large values of η, the value of Λ given by (5.13) is greater than that given by (5.14). Thus, the curves must intersect for at least one real positive value of η. At the first point of intersection, the values of Λ given by (5.13) and (5.14), are, of course, equal. Also, as η increases the ratio $[\sinh(2\lambda^{1/2}\cosh\delta)\eta]/[\sinh(2\lambda^{1/2}\sinh\delta)\eta]$ increases monotonically. Consequently, no further intersection is possible and we conclude that equation (5.12) has one and only one real positive solution corresponding to the conditions $\lambda>1$ and (5.18).

Turning now to the equation (5.11) for the barreling mode, we note that its solution for η is given by the intersection of the two curves

$$\Lambda = \frac{1}{\cosh\delta}\,[-(1-\lambda)^2 + 4\cosh^2\delta]\sinh(2\lambda^{1/2}\cosh\delta)\eta \tag{5.21}$$

and

$$\Lambda = -\frac{1}{\sinh\delta}\,[(1+\lambda)^2 + 4\sinh^2\delta]\sinh(2\lambda^{1/2}\sinh\delta)\eta. \tag{5.22}$$

If $\lambda > 1$, so that (5.20) is valid, we see that if

$$-(1-\lambda)^2 + 4\cosh^2\delta \geq 0, \tag{5.23}$$

the value of Λ given by (5.21) is non-negative for all positive η, while that given by (5.22) is negative for all positive η. Consequently, if $\lambda > 1$ and (5.23) is satisfied, equation (5.11) has no real solution for η.

We now consider the case when $\lambda > 1$, so that (5.20) is valid, but

$$-(1-\lambda)^2 + 4\cosh^2\delta < 0. \tag{5.24}$$

Now, since $A > -1$, it is evident that

$$-(1-\lambda)^2 + 4\cosh^2\delta > -[(1+\lambda)^2 + 4\sinh^2\delta]. \tag{5.25}$$

Therefore, the slope of (5.21) at $\eta = 0$ is greater than that of (5.22). However, in view of (5.16) and (5.24), for η large, the value of Λ given by (5.21) is less than that given by (5.22). Consequently, the curves (5.21) and (5.22) must intersect at least once for positive real η. That they have only one intersection can be seen from the following consideration. At the first intersection, the values of Λ given by (5.21) and (5.22) are equal. Also, as η increases, the ratio $[\sinh(2\lambda^{1/2}\cosh\delta)\eta]/[\sinh(2\lambda^{1/2}\sinh\delta)\eta]$ increases monotonically. Consequently, no further intersection is possible and we conclude that equation (5.11) has one and only one solution corresponding to the conditions $\lambda > 1$ and (5.24).

If $\lambda \leq 1$, so that (5.20) is violated, then $[-(1-\lambda)^2 + 4\cosh^2\delta]$ is necessarily positive. The value of Λ given by (5.21) is positive for all positive η, while that given by (5.22) is negative for all positive η. Consequently, if $\lambda \leq 1$, equation (5.11) has no real positive solution for η.

We may summarize the conclusion reached so far in this section. We introduce the notation

$$\theta = -(1-\lambda)^2 + 4\cosh^2\delta, \tag{5.26}$$

where, cf. $(5.7)_1$ and (4.5),

$$\cosh^2\delta = \frac{1}{4\lambda}\,[(1+\lambda)^2 + A(1-\lambda)^2] \tag{5.27}$$

and

$$A = \frac{2(\lambda_1+\lambda_2)^2}{W_1 + \lambda_3^2 W_2}\,(W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22}). \tag{5.28}$$

Under the condition that $A > -1$, we have found that

(i) if $\lambda \leq 1$ (tensile conditions), no real positive value of η can be found for which either a critical flexural or a critical barreling configuration can exist;
(ii) for any specified $\lambda > 1$, there exists one, and only one, value of η for which a critical flexural configuration can exist provided that $\theta > 0$ and no such values of η if $\theta \leq 0$;
(iii) for any specified $\lambda > 1$, there exists one and only one value of η for which a critical barreling configuration can exist provided that $\theta < 0$ and no such values of η if $\theta \geq 0$.

We note that, whatever A may be, the condition $\theta > 0$ is satisfied for $\lambda = 1$. Corresponding to this value of λ, we see, from (5.12), that $\eta = 0$, i.e. a critical flexural configuration can exist in the limiting case $l_2 = 0$. Assuming that A is a continuous function of λ, the solution of (5.12) for η is a continuous function of λ in the range $\theta > 0$. Thus, there exists some range of values of λ from 1 to some value greater than 1, such that, for each value of λ in this range, there is one value of η for which a critical flexural configuration exists. We note, also, from (5.12), that as $\theta \to 0$ from above, $\eta \to \infty$ from below. Thus, since A and therefore θ is a continuous function of λ, for each value of η in the range 0 to ∞, there exists at least one value of λ corresponding to a critical flexural configuration, provided A does not increase so rapidly with λ that θ is non-zero for all $\lambda > 1$. We note that this latter qualification cannot be operative if A is independent of λ. If it is operative, then there may be values of η for which no critical flexural configuration can exist for any value of λ. The impossibility of critical flexural configurations corresponding to sufficiently large values of η may also be realized for other reasons. For example, the material may fracture before the value of λ, which would otherwise correspond to such a configuration, is reached.

6. DISCUSSION OF THE CASE WHEN $\Omega_1^2 = \Omega_2^2$

From (4.8), we see that $\Omega_1^2 = \Omega_2^2$ if and only if

$$(1-\lambda)^2(A+1)\{A(1-\lambda)^2 + (1+\lambda)^2\} = 0. \tag{6.1}$$

Now, (6.1) is satisfied if

$$\lambda = 1, \quad \text{or} \quad A = -1, \quad \text{or} \quad A = -\left(\frac{1+\lambda}{1-\lambda}\right)^2. \tag{6.2}$$

If $(6.2)_3$ applies, it follows from (4.8) that

$$\Omega_1^2 = \Omega_2^2 = -\lambda\Omega^2, \tag{6.3}$$

i.e. Ω_1^2 and Ω_2^2 are negative. In accordance with the discussion at the end of Section 4, we omit further consideration of this case.

If $(6.2)_2$ applies, the general solution of (4.4) is

$$U_2 = L_1 \cosh \hat{\Omega}\xi_2 + L_2 \sinh \hat{\Omega}\xi_2 + M_1\xi_2 \sinh \hat{\Omega}\xi_2 + M_2\xi_2 \cosh \hat{\Omega}\xi_2, \tag{6.4}$$

where

$$\hat{\Omega} = \Omega\lambda^{1/2}. \tag{6.5}$$

Since $A = -1$, we have from (4.16),

$$B = \lambda^2 + 2\lambda. \tag{6.6}$$

Introducing (6.4) and (6.6) into the boundary conditions (4.14) and (4.15), we obtain

$$\begin{aligned}\hat{\Omega}(\lambda+1)[L_1\cosh\hat{\Omega}l_2 + M_1 l_2\sinh\hat{\Omega}l_2 \pm (L_2\sinh\hat{\Omega}l_2 + M_2 l_2\cosh\hat{\Omega}l_2)] \\ + 2(M_1\cosh\hat{\Omega}l_2 \pm M_2\sinh\hat{\Omega}l_2) = 0, \\ \hat{\Omega}(\lambda+1)[L_1\sinh\hat{\Omega}l_2 + M_1 l_2\cosh\hat{\Omega}l_2 \pm (L_2\cosh\hat{\Omega}l_2 + M_2 l_2\sinh\hat{\Omega}l_2)] \\ + (\lambda-1)(M_1\sinh\hat{\Omega}l_2 \pm M_2\cosh\hat{\Omega}l_2) = 0.\end{aligned} \tag{6.7}$$

By an argument similar to that used in passing from equations (4.18) to (4.20) and (4.21), we find, from (6.7), that

$$L_1 = M_1 = 0 \quad \text{and} \quad \frac{\sinh 2\hat{\Omega}l_2}{2\hat{\Omega}} = -l_2\frac{1+\lambda}{3-\lambda}$$

or (6.8)

$$L_2 = M_2 = 0 \quad \text{and} \quad \frac{\sinh 2\hat{\Omega}l_2}{2\hat{\Omega}} = l_2\frac{1+\lambda}{3-\lambda},$$

unless $\lambda = 3$. We note, from (5.26) and (5.27), that when $A = -1$ and $\lambda = 3$, $\theta = 0$, i.e. $\lambda = 3$ is the value of λ which separates the ranges in which critical flexural and critical barreling conditions obtain. With $(5.2)_1$ and (6.5), we can rewrite (6.8) as

$$\frac{\sinh 2\eta\lambda^{1/2}}{2\eta\lambda^{1/2}} = \pm\frac{1+\lambda}{3-\lambda}, \tag{6.9}$$

the upper sign corresponding to flexural conditions ($\lambda < 3$) and the lower sign to barreling conditions ($\lambda > 3$). It is evident that (6.9) has no real solutions for η if $\lambda < 1$.

We remark that an equation analogous to (6.9) was previously derived by Guo Zhong-Heng[5] in his discussion of the instability of a circular plate under uniform radial thrust.

7. NUMERICAL RESULTS

If A is assigned some constant value, (5.11) and (5.12) become equations for the determination of η as a function of λ. The special case of $A = -1$ is covered by (6.9). The resulting values of η are plotted against λ in Fig. 1 for $A = -1, 0, 11/9$ and for λ in the range from 2·5 to 4·5. The curve for $A = 0$ (which applies to the Mooney–Rivlin material) is plotted over a wider range of λ in Fig. 2. In Figs. 1 and 2, the full lines correspond to the values of λ given by (5.12) and thus pertain to critical flexural conditions. The broken lines correspond to the values of λ given by (5.11) and thus pertain to critical barreling conditions. The curve for $A = -1$ has an asymptote at $\lambda = 3$ and that for $A = 11/9$ has an asymptote at $\lambda = 4$.

We see, as has already been shown in Section 5, that, for any specified value of A, the ranges of λ which apply to critical flexural and barreling conditions do not overlap. The values of λ separating these ranges are, of course, given by $\theta = 0$, where θ is defined by (5.26) and (5.27); i.e. they are given by the equation

$$\lambda^3 - (3+A)\lambda^2 + (2A-1)^2 - (1+A) = 0. \tag{7.1}$$

It can easily be shown that, if A is constant, equation (7.1) has only one solution for λ which is greater than unity. This is plotted against A in Fig. 3.

We see from Figs. 1 and 2 that, for any given value of η, the critical value of λ for flexure is lower than that for barreling and, accordingly, under conditions in which the loading is

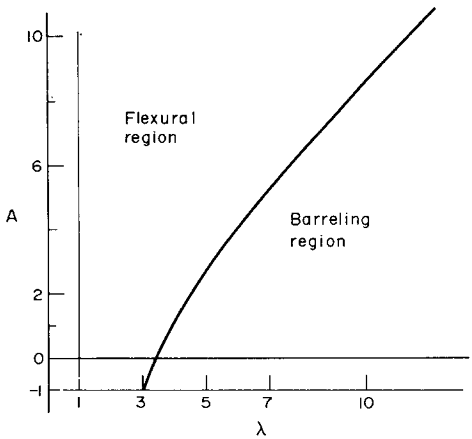

Fig. 3. Relation between A and values of λ separating flexural and barreling regimes [equation (7.1)].

increased from zero, flexural instability will occur before the load appropriate to barreling is reached.

The values of η obtained from (5.11) for constant values of A approach zero as λ becomes infinitely large. To see this, we employ (5.7) and rewrite (5.11) in the form

$$\frac{\sinh\left[(\lambda+1)\left\{1+A\left(\dfrac{\lambda-1}{\lambda+1}\right)^2\right\}^{1/2}\eta\right]}{\sinh[(\lambda-1)\{1+A\}^{1/2}\eta]} = \frac{\lambda+1}{\lambda-1}\,\frac{\left\{1+A\left(\dfrac{\lambda-1}{\lambda+1}\right)^2\right\}^{1/2}}{\{1+A\}^{1/2}}\,\frac{1+\dfrac{1}{\lambda}(A+1)\left(\dfrac{\lambda-1}{\lambda+1}\right)^2}{\left(\dfrac{\lambda-1}{\lambda+1}\right)^2-\dfrac{1}{\lambda}\left\{1+A\left(\dfrac{\lambda-1}{\lambda+1}\right)^2\right\}}. \tag{7.2}$$

Now for any constant $A > -1$, the right member of (7.2) is seen to approach unity as $\lambda \to \infty$. If η does not approach zero as $\lambda \to \infty$ then the left member of (7.2) approaches

$$\sinh[(\lambda+1)(1+A)^{1/2}\eta]/\sinh[(\lambda-1)(1+A)^{1/2}\eta] \approx \exp[2(1+A)^{1/2}\eta]$$

which is greater than unity. Hence, we conclude that $\eta \to 0$. This result is in apparent conflict with that obtained by Nowinski[9] who dealt with a neo-Hookean material for which $A = 0$.

From curves similar to those shown in Figs. 1 and 2, we can also obtain the critical values of λ for flexure and barreling for any given η, even when A is not independent of λ. To do this we construct a series of curves, as in Fig. 1, over a wider range of λ and for various constant values of A. From these, pairs of points (A, λ), corresponding to flexure and barreling, are read off for some specified value of η. These pairs are then plotted against λ and the relevant curve of A vs λ is drawn on the same graph. The intersections of the former curves with the latter then give the critical values of λ for flexure and barreling corresponding to the value of η for which the curves are drawn. It may, of course, happen that the A vs λ curve does not intersect the other curves, in which case instability could not occur for that particular value of η.

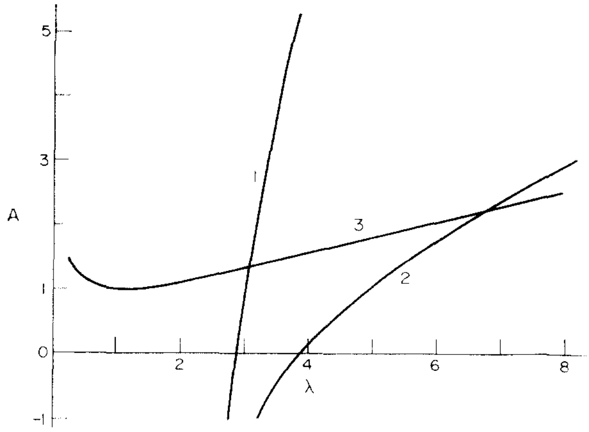

Fig. 4. Illustration of the method for determining critical values of λ when A is nonconstant.

This procedure has been carried out for $\eta = 1{\cdot}5$ and the resulting curves, labeled 1 and 2, are shown in Fig. 4. These correspond to critical flexural and barreling conditions, respectively. Curve 3 in Fig. 4 has been obtained by taking

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) + \tfrac{1}{16}(C_1 + C_2)(I_2 - 3)^2, \tag{7.3}$$

which, with $\lambda_3 = 1$ and $\lambda_1\lambda_2 = \lambda_1^2\,\lambda = 1$, leads to

$$A = \frac{1}{4\lambda}(1 + \lambda)^2. \tag{7.4}$$

From Fig. 4 we see that the critical values of λ for flexure and barreling in this case are approximately 3·1 and 6·8 respectively.

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Абстракт — Толстая прямоугольная плита из несжимаемого изотропного, упругого материала находится под влиянием однородной деформации, вызванной растягивающими силами или давлением, приложенными к паре противоположенных торцов. Применяется теория малых деформаций, наложенных на конечные деформации, с целью определения критических условий, для которых могут существовать решения разветления (т. е. смежные позиции равновесия). Исследуются такие же смежные позиции равновесия, для которых наложенная деформация двухмерна и копланирна с усилием от нагрузки и направлением толщины плиты, причем торцы плиты нормальны к ее толщиней свободны от усилий. Определяется некоторое число теорем, по отношению к критическим условиям для наложенных деформаций при изгибе или осаживанию, учитывая более общие условия функции энергии деформаций по сравнению с такими же использованными в предидущей работе. Указано, также, как можно эти результаты применить для определения условий разветления, которые соответствуют к любой требованной функции энергии деформации.